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Relative length of longest paths and longest cycles in triangle-free graphs

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Abstract

In this paper, we study triangle-free graphs. Let $G=(V,E)$ be an arbitrary triangle-free graph with minimum degree at least two and $\sigma_4(G) \geqslant |V(G)|+2$. We first show that either for any path P in G there exists a cycle C such that $|V_P \backslash V_C| \leqslant 1$, or G is isomorphic to exactly one exception. Using this result, we show that for any set S of at most δ vertices in G there is a cycle C such that $S \subseteq V_C$.

Keywords: Triangle-free graph; Cycle; Ore-condition; Relative length

1. Introduction

Let $G=(V_G,E_G)$ be a graph of order $|V_G|=n$. For graph terminology not defined below we refer to [10]. For simplicity, we sometimes denote $|V_G|$ by $|G|$ and "$u \in V_G$" by "$u \in G$". For a vertex $u \in G$, we denote its neighborhood by $N_G(u)=\{v \mid uv \in E_G\}$. The degree of a vertex $u \in V_G$ is denoted by $d_G(u)$ and the minimum degree of G is denoted by δ_G. If no confusion is possible we will omit the subscript G in the above (and later) notations. We denoted by $H \subseteq G$ that H is a subgraph of G.

Let $P=v_1v_2\ldots v_p$ be some path of order p. Vertices u and v are called the *ends* of P. The order of a longest path in G is denoted by p_G. A vertex u is called a *cut vertex* of a connected graph G if $G[V\backslash\{u\}]$ is disconnected. A graph $G=(V,E)$ is called k-*connected* if $G[V\backslash U]$ is connected for any set $U \subseteq V$ of at most $k-1$ vertices. A *cycle* C is a sequence $v_1v_2\ldots v_pv_1$ of distinct vertices, where each pair of consecutive vertices forms an edge. The order of a longest cycle in a graph G is called the *circumference* c_G. A cycle C is called *dominating* if $G-C$ is edgeless.

Let $G=(V,E)$ be a graph. A set $U \subseteq V$ is called *independent* if G does not contain edges with both end vertices in U. The number of vertices in a maximum independent set is called the *independence number* of G. We denote

$$\sigma_k(G)=\min\left\{\sum_{i=1}^{k} d_G(x_i) \middle| x_1,x_2,\ldots,x_k \text{ are distinct and independent}\right\}.$$

If the independence number of G is less than k, then we define $\sigma_k(G)=\infty$.

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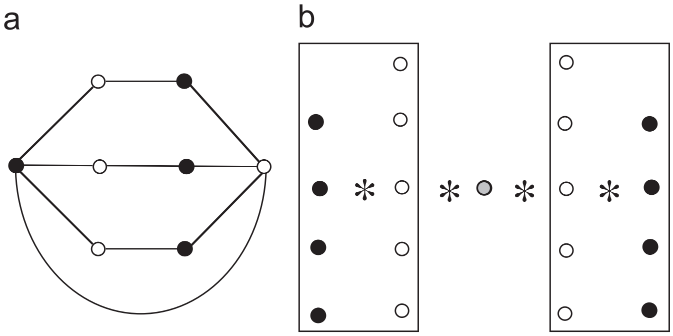

Fig. 1. (a) exception for Theorem 3, (b) the graph H_5.

A graph G is called *hamiltonian* if G contains a cycle C with $V_C = V_G$. See Gould [14] for a survey on hamiltonian graphs. For a graph G that is not hamiltonian, a natural question is to ask how close it is to hamiltonicity. To measure this, we can take the difference $p - c$ between the order of a longest path and the circumference of G. We observe that $p - c = 0$ if and only if G is hamiltonian. Furthermore, $p - c \leqslant 1$ implies that all longest cycles are dominating. In order to see this, suppose C is a non-dominating longest cycle of a 2-connected graph G. So $|C| = c$. Since C is non-dominating, $G - C$ contains an edge. We take a shortest path connecting this edge to C and extend it with $|C| - 1$ edges of C. We call the resulting path P. We then find that $p - c \geqslant |P| - |C| \geqslant 2$, a contradiction. In the literature many results on dominating cycles and the *relative length* $p - c$ of longest paths and cycles can be found (see, e.g., [16,18,22,23]).

Ore [19] showed that a graph G with $\sigma_2 \geqslant n$ is hamiltonian. Bondy [5] studied σ_3 and proved the following result.

Theorem 1 (*Bondy [5]*). *If G is a 2-connected graph with $\sigma_3 \geqslant n + 2$, then all longest cycles are dominating.*

Enomoto et al. [12] proved the following.

Theorem 2 (*Enomoto et al. [12]*). *If G is a 2-connected graph with $\sigma_3 \geqslant n + 2$, then $p - c \leqslant 1$.*

We already noted that $p - c \leqslant 1$ implies that all longest cycles are dominating. Clearly, the opposite is not true. Hence, Theorem 2 generalizes Theorem 1.

In this paper we are interested in proving a similar result for *triangle-free* graphs (graphs that do not contain an induced K_3) corresponding to Theorem 2 of Enomoto et al. [12]. Is it possible to make a jump from σ_3 to σ_4 when we restrict ourselves to this graph class? Triangle-free graphs are the natural generalization of bipartite graphs and therefore have been widely studied in the literature, also in the context of hamiltonian research (cf. [2,3,7,13,17]). Broersma et al. [9] showed that a 2-connected triangle-free graph with $\sigma_3 \geqslant (n + 5)/2$ contains a longest cycle that is dominating. The lower bound on σ_3 is tight, even for the existence of dominating cycles. Note that graphs satisfying the conditions of this theorem might contain longest cycles that are not dominating. However, if $\sigma_2 \geqslant (n + 1)/2$, then all longest cycles are dominating [24]. This lower bound is almost best possible by examples due to Ash and Jackson [1].

The main result of this paper is as follows. Its proof is given in Section 2.

Theorem 3. *Let G be a triangle-free graph with $\delta \geqslant 2$ not isomorphic to the graph in Fig. 1(a). If $\sigma_4 \geqslant n + 2$ then for any path P there exists a cycle C such that $|P - C| \leqslant 1$.*

We note that Theorem 3 immediately implies that $p - c \leqslant 1$. Hence, this result for triangle-free graphs is "similar" to Theorem 2 of Enomoto et al. for 2-connected graphs.

The lower bound on σ_4 in Theorem 3 is tight. In order to see this, consider the graphs $H_k = \overline{K_{k-1}} * \overline{K_k} * K_1 * \overline{K_k} * \overline{K_{k-1}}$ of order $n = 4k - 1$ for $k \geqslant 2$. For an illustration of the case $k = 5$, see Fig. 1(b). Obviously, each H_k is triangle-free. It is easy to check that each H_k has minimum degree $2 \leqslant \delta_{H_k} = k = (n + 1)/4$. Since each H_k contains at least four vertices of minimum degree, we find that $\sigma_4(H_k) = n + 1$. Furthermore, each H_k contains a path P of order $|P| = n$. However, any cycle can pass through K_1 at most once. So a longest cycle C contains all vertices of exactly one $\overline{K_{k-1}}$, one adjacent $\overline{K_k}$ and the vertex of the K_1. Hence, for all $k \geqslant 2$, the circumference of H_k is $c_{H_k} = 2k = (n+1)/2 \leqslant n-2$. So, for P there does not exist a cycle C with $|P - C| \leqslant 1$. This means that the bound on σ_4 is tight indeed.

In Theorem 3 no condition is imposed on the connectivity of a graph. A natural question (cf. Theorem 2) is to ask whether adding such a condition would be helpful for decreasing the lower bound on σ_4. However, this is not the case: we can add all possible edges between the left $\overline{K_{k-1}}$ and the right $\overline{K_{k-1}}$ in H_k. This way we obtain a new graph H'_k that is still triangle-free, has minimum degree $(n+1)/4 \geqslant 2$ and $\sigma_4(H'_k) = n + 1$, and furthermore contains a path of length n. However, a longest cycle C will pass through all vertices except one vertex of each $\overline{K_{k-1}}$, so $|C| = c_{H'_k} = n - 2$. We reach the same conclusion as before.

In the literature the following related problem has been studied for general graphs and graph classes (see, e.g., [4,6,8,11,15,20,21]): for a given graph G, does any subset S of vertices of restricted size have some cycle passing through it? As an application of Theorem 3, we obtain the following result for triangle-free graphs. Its full proof is given in Section 3.

Theorem 4. *Let G be a triangle-free graph with $\delta \geqslant 2$. If $\sigma_4 \geqslant n + 2$, then for any set S of at most δ vertices, there exists a cycle C such that $S \subseteq V_C$.*

This result implies that a triangle-free graph with $\delta \geqslant 2$ and $\sigma_4 \geqslant n+2$ is 2-connected. On the other hand, the previously defined graphs H_k contain a cut vertex, namely the vertex of the K_1. Hence, the lower bound on σ_4 in Theorem 4 is tight. In Section 3 we show that a triangle-free graph with $\delta \geqslant 2$ and $\sigma_4 \geqslant n + 1$ is connected. The lower bound on σ_4 is tight due to the graphs $K_{k,k} \cup K_{k,k}$ for $k \geqslant 2$.

We finish the section by introducing some additional notations. Let $G = (V, E)$ be a graph. For a subset $U \subseteq V$ and vertex $u \in V$ we sometimes write "$U \backslash u$" instead of "$U \backslash \{u\}$".

Let H be a subgraph of G. We denote $N_G(x) \cap V_H$ by $N_H(x)$ and its cardinality $|N_H(x)|$ by $d_H(x)$. The set of neighbors $\bigcup_{v \in H} N_G(v) \backslash V_H$ is denoted by $N_G(H)$ or $N(H)$. For an edge $e = uv$ in G, we write $N(e) = N(\{u, v\})$. For a subgraph $F \subseteq G$, we write $N_G(H) \cap V_F$ as $N_F(H)$.

Let $C = v_1 v_2 \ldots v_p v_1$ be a cycle with a fixed orientation. The successor v_{i+1} of v_i is denoted by v_i^+ and its predecessor v_{i-1} by v_i^-. For a vertex subset A in C, we denote $\{v_i^+ | v_i \in A\}$ and $\{v_i^- | v_i \in A\}$ by A^+ and A^-, respectively. The segment $v_i v_{i+1} \ldots v_j$ is written as $v_i \overrightarrow{C} v_j$, where the subscripts are to be taken modulo $|C|$. The converse segment $v_j v_{j-1} \ldots v_i$ is written as $v_j \overleftarrow{C} v_i$. Similarly, for a path $P = u_1 u_2 \ldots u_p$, we use the notations $u_i \overrightarrow{P} u_j = u_i u_{i+1} \ldots u_j$ and $u_j \overleftarrow{P} u_i = u_j u_{j-1} \ldots u_i$.

2. The proof of Theorem 3

Let S be a vertex subset of G. If a path P is a longest path over all paths containing S, then we call P a *maximal path for S*. The set of all maximal paths for S is denoted by $\mathscr{P}(S)$. Before proving Theorem 3 we first show the following lemma.

Lemma 5. *Let G be a triangle-free graph with $\delta_G \geqslant 2$ not isomorphic to the graph in Fig. 1(a). Then for any path R, there either exists a path in $\mathscr{P}(V_R)$ such that the degree sum of the ends is at least $\sigma_4(G)/2$, or else a cycle C such that $|R - C| \leqslant 1$.*

Proof. Let G be a triangle-free graph with $\delta_G \geqslant 2$. Assume that G is not isomorphic to the graph in Fig. 1(a). Let R be any path in G and $P = u_1 u_2 \ldots u_p \in \mathscr{P}(V_R)$ such that the degree sum of the ends is maximal in $\mathscr{P}(V_R)$. Notice that $N(u_1) = N_P(u_1)$ and $N(u_p) = N_P(u_p)$. So all neighbors of u_1 and u_p in G belong to P.

Suppose there are vertices $u_i \in N(u_1) \backslash u_2$ and $u_j \in N(u_p) \backslash u_{p-1}$ such that $i \leqslant j$. Then $\{u_1, u_{i-1}, u_{j+1}, u_p\}$ is independent; otherwise there is a triangle (forbidden) or a cycle containing V_R (we are done). Because $d(u_1) + d(u_{i-1}) + d(u_{j+1}) + d(u_p) \geqslant \sigma_4$, one of the degree sums $d(u_1) + d(u_p)$ and $d(u_{i-1}) + d(u_{j+1})$ is at least $\sigma_4/2$. Hence, at least one of the paths P or $u_{i-1} \overleftarrow{P} u_1 u_i \overrightarrow{P} u_j u_p \overleftarrow{P} u_{j+1}$ is a desired path.

In the remaining case we have

$$i > j \text{ for any two vertices } u_i \in N(u_1)\backslash u_2 \text{ and } u_j \in N(u_p)\backslash u_{p-1}. \tag{1}$$

Suppose there is a vertex $u_s \in N_P(u_1)\backslash\{u_2, u_{p-2}\}$. Since $\delta_G \geqslant 2$ and $N(u_p) = N_P(u_p)$, vertex u_p has a neighbor $u_t \neq u_{p-1}$ on P. Then we find that the path $P' = u_{t+1}\overrightarrow{P}u_s u_1 \overrightarrow{P} u_t u_p \overleftarrow{P} u_{s+1}$ is a path in $\mathscr{P}(V_R)$. The vertex u_1 is not adjacent to u_{t+1} nor u_{s+1}; otherwise there is a triangle or a cycle containing V_R. Also, the vertex u_p is not adjacent to u_{t+1} nor to u_{s+1} by statement (1) and $u_s \neq u_{p-2}$. Thus $\{u_1, u_{t+1}, u_{s+1}, u_p\}$ is an independent set. Hence, at least one of the paths P and P' is a desired path as in the previous case. Therefore $N(u_1) = \{u_2, u_{p-2}\}$ and, by symmetry, $N(u_p) = \{u_3, u_{p-1}\}$. Furthermore, by the maximality of the degree sum of the ends of P we deduce that

$$\text{the degree of an end of any path in } \mathscr{P}(V_R) \text{ is two.}$$

Because the path $u_1u_2u_3u_p\overleftarrow{P}u_4$ is in $\mathscr{P}(V_R)$, the vertex u_1 has to be adjacent to $u_4^{++} = u_6$; otherwise, as in the above case, we can obtain a desired cycle or path. Therefore $u_6 = u_{p-2}$, i.e., $p = 8$, and so any vertex in $\{u_1, u_2, u_4, u_5, u_7, u_8\}$ is the end of some path in $\mathscr{P}(V_R)$, and consequently has degree two. As G is triangle-free, the vertices u_1, u_5 and u_7 are mutually disjoint. If $G - P$ is not empty, then for any $x \in G - P$, the set $\{x, u_1, u_5, u_7\}$ is independent. Hence we find that

$$d(x) \geqslant \sigma_4 - (d(u_1) + d(u_5) + d(u_7)) \geqslant n + 2 - 6 = n - 4.$$

However, x is adjacent to none of the vertices in $\{u_1, u_2, u_4, u_5, u_7, u_8\}$ because their degrees are all equal to two. Thus $d(x) \leqslant n - 7$, a contradiction. Therefore $G - P = \emptyset$ and $n = 8$. As u_3 is adjacent to none of the vertices u_1, u_5, u_7, vertex u_3 has to be adjacent to u_6; otherwise $d(u_1) + d(u_3) + d(u_5) + d(u_7) = 9 < n + 2$. Hence G is isomorphic to the graph in Fig. 1(a), a contradiction. □

We are ready to prove Theorem 3. Let G be a triangle-free graph with $\delta \geqslant 2$ and $\sigma_4 \geqslant n + 2$ that is not isomorphic to the graph in Fig. 1(a). Let R be any path in G. We prove that G contains a desired cycle, i.e., a cycle C such that $|R - C| \leqslant 1$.

Suppose the independence number of G is at most three. Then $\sigma_4(G) = \infty$. By Lemma 5, there exists a cycle C such that $|R - C| \leqslant 1$.

From now on we assume that the independence number of G is at least four. Let $P = u_1u_2 \ldots u_p \in \mathscr{P}(V_R)$ such that

$$\text{the degree sum of the ends is maximal in } \mathscr{P}(V_R). \tag{2}$$

Then from Lemma 5, $d(u_1) + d(u_p) \geqslant \sigma_4/2$. Notice that we may assume that there is no path in $\mathscr{P}(V_R)$ whose ends are adjacent; otherwise obviously there exists a cycle containing V_R.

If there is $u_l \in N_P(u_1) \cap N_P(u_p)^+$, then the cycle $u_1\overrightarrow{P}u_l^- u_p \overleftarrow{P} u_l u_1$ is a desired cycle. Thus we can suppose $N_P(u_1) \cap N_P(u_p)^+ = \emptyset$. Similarly, we get $N_P(u_1) \cap N_P(u_p)^{++} = \emptyset$ and $N_P(u_1)^- \cap N_P(u_p)^+ = \emptyset$. If $N_P(u_1)^- \cap N_P(u_p)^{++}$ is also empty, then $N_P(u_1)$, $N_P(u_1)^-$, $N_P(u_p)^+$ and $(N_P(u_p)\backslash u_p)^{++}$ are mutually disjoint. Hence we find that

$$\begin{aligned} n \geqslant |P| &\geqslant |N_P(u_1)| + |N_P(u_1)^-| + |N_P(u_p)^+| + |(N_P(u_p)\backslash u_p)^{++}| \\ &\geqslant 2d(u_1) + 2d(u_p) - 1 \geqslant \sigma_4 - 1 > n. \end{aligned}$$

This is a contradiction. Therefore $N_P(u_1)^- \cap N_P(u_p)^{++} \neq \emptyset$.

Let $u_i \in N_P(u_1)^- \cap N_P(u_p)^{++}$.

Claim 1. *If $d(u_i) + d(u_{i-1}) > n/2$, then there is a desired cycle.*

Proof. Let $e_0 = x_1x_2 = u_{i-1}u_i$ and

$$C = u_1\overrightarrow{P}u_{i-2}u_p\overleftarrow{P}u_{i+1}u_1 = v_1v_2 \ldots v_{p-2}v_1$$

which occur on C in the order of their indices. Notice that $N(e_0) = N(x_1) \cup N(x_2)\backslash\{x_1, x_2\} \subset V_C$ because P is a maximal path for V_R.

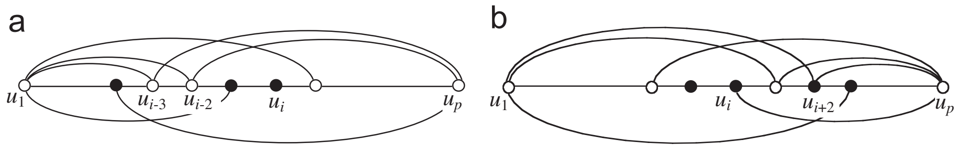

Fig. 2.

If $N(e_0)$ and $N(e_0)^+$ are not disjoint, then there exists a triangle or a desired cycle. Hence $N(e_0) \cap N(e_0)^+ = \emptyset$. In the set of segments $C - N(e_0)$, there are two segments $v_s^+ \overrightarrow{C} v_{s'}^-$ and $v_t^+ \overrightarrow{C} v_{t'}^-$ such that $\{v_s, v_{t'}\} \subset N(x_1)$ and $\{v_{s'}, v_t\} \subset N(x_2)$. Then $v_{s+2}, v_{t+2} \notin N_C(e_0) \cup N_C(e_0)^+$; otherwise there is a desired cycle. Therefore, we find

$$\begin{aligned} n-2 &\geqslant |C| \geqslant |N(e_0)| + |N(e_0)^+| + |\{v_{s+2}, v_{t+2}\}| \\ &= |N_C(x_1)| + |N_C(x_1)^+| + |N_C(x_2)| + |N_C(x_2)^+| + |\{v_{s+2}, v_{t+2}\}| \\ &= 2(d(x_1) - 1) + 2(d(x_2) - 1) + 2 = 2(d(x_1) + d(x_2)) - 2 > n - 2. \end{aligned}$$

This is a contradiction. □

If $\delta \geqslant (n+2)/4$, then our proof is completed now by this claim. We divide our argument into two cases.

Case 1. $|N_P(u_1)^- \cap N_P(u_p)^{++}| = 1$: Let $\{u_i\} = N_P(u_1)^- \cap N_P(u_p)^{++}$. We show that $d(u_i) + d(u_{i-1}) > n/2$. Because

$$\begin{aligned} n \geqslant |P| &\geqslant |N_P(u_1)| + |N_P(u_1)^-| + |N_P(u_p)^+| + |(N_P(u_p) \backslash u_{p-1})^{++}| \\ &\quad - |N_P(u_1)^- \cap N_P(u_p)^{++}| \\ &= 2d(u_1) + 2d(u_p) - 1 - 1 \geqslant \sigma_4 - 2 \geqslant n, \end{aligned}$$

it holds that

$$V_G = V_P = N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \backslash u_{p-1})^{++} \tag{3}$$

and that

$$d(u_1) + d(u_p) = \tfrac{n}{2} + 1. \tag{4}$$

Hence the order n is even.

Because

$$u_{i-3} \overleftarrow{P} u_1 u_{i+1} u_i u_{i-1} u_{i-2} u_p \overleftarrow{P} u_{i+2} \in \mathscr{P}(V_R),$$

we have $u_{i-3}u_{i+2} \notin E_G$. If $u_{i-3}u_1 \in E_G$ then

$$u_{i-2} \notin N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \backslash u_{p-1})^{++}.$$

See Fig. 2(a). This contradicts (3). Thus $u_{i-3}u_1 \notin E_G$. Especially, u_{i-3} is not u_2. Similarly, if $u_{i+2}u_p \in E_G$, then

$$u_{i+2} \notin N_P(u_1) \cup N_P(u_1)^- \cup N_P(u_p)^+ \cup (N_P(u_p) \backslash u_{p-1})^{++}.$$

See Fig. 2(b). This also contradicts (3). Hence, $u_{i+2}u_p \notin E_G$ and especially $u_{i+2} \neq u_{p-1}$. As $u_1u_p \notin E_G$, $\{u_1, u_{i-3}, u_{i+2}, u_p\}$ is an independent set.

Let $x_1x_2 = u_{i-1}u_i$ and $w_1 = u_{i-3}$ and $w_2 = u_{i+2}$. Because $d(u_1) + d(u_p) + d(w_1) + d(w_2) \geqslant \sigma_4 \geqslant n+2$, we have

$$d(w_1) + d(w_2) = \tfrac{n}{2} + 1$$

by (2) and (4). Notice that none of u_1, u_p, w_1, w_2 are adjacent to x_1 nor x_2; otherwise easily we can find a triangle or a desired cycle. Hence for each i, j,

$$d(u_1) + d(u_p) + d(x_i) + d(w_j) \geqslant n + 2.$$

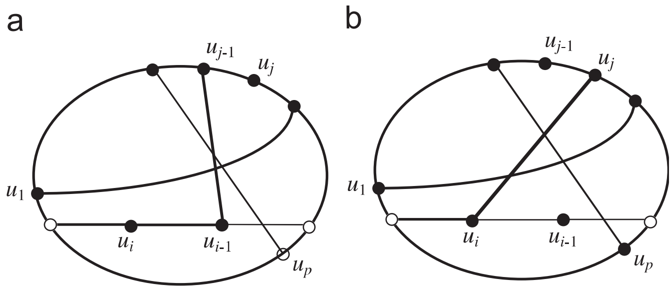

Fig. 3.

Assume that $n/2$ is even, say $2l$. Then $d(u_1)+d(u_p)=d(w_1)+d(w_2)=2l+1$. By symmetry, we can suppose that $d(w_1)\leqslant l$. Because

$$d(u_1)+d(u_p)+d(x_i)+d(w_1)\geqslant 4l+2,$$

we have $d(x_i)\geqslant l+1$ for $i=1,2$. Hence $d(x_1)+d(x_2)\geqslant 2l+2>n/2$.

Suppose $n/2$ is odd, say $2l+1$. Then $d(u_1)+d(u_p)=d(w_1)+d(w_2)=2l+2$. By symmetry, we may assume that $d(w_1)\leqslant l+1$. Because

$$d(u_1)+d(u_2)+d(w_1)+d(x_i)\geqslant 4l+4,$$

we have $d(x_i)\geqslant l+1$ for $i=1,2$. Thus $d(x_1)+d(x_2)\geqslant 2l+2>n/2$.

Therefore, in either cases, $d(u_i)+d(u_{i-1})>n/2$, and hence we are done by Claim 1.

Case 2: $|N_P(u_1)^-\cap N_P(u_p)^{++}|\geqslant 2$: Let $u_i,u_j\in N_P(u_1)^-\cap N_P(u_p)^{++}$ $(i>j)$. If u_{i-1} is adjacent to u_{j-1}, then the cycle $u_1\overrightarrow{P}u_{j-1}u_{i-1}u_iu_i^+\overrightarrow{P}u_pu_{i-2}\overleftarrow{P}u_j^+u_1$ is a desired cycle. See Fig. 3(a). Therefore $u_{i-1}u_{j-1}\notin E_G$. Similarly we can obtain $u_iu_j\notin E_G$, see Fig. 3(b). Hence we find that

$$\begin{aligned}&(d(u_1)+d(u_p)+d(u_{i-1})+d(u_{j-1}))+(d(u_1)+d(u_p)+d(u_i)+d(u_j))\\&\quad\geqslant\sigma_4+\sigma_4\geqslant 2n+4.\end{aligned}$$

By symmetry, we may without loss of generality assume that

$$d(u_1)+d(u_p)+d(u_{i-1})+d(u_i)\geqslant n+2. \tag{5}$$

Let $e_0=x_1x_2=u_{i-1}u_i$ and C be the cycle $u_1\overrightarrow{P}u_{i-2}u_p\overleftarrow{P}u_{i+1}u_1=v_1v_2\ldots v_{p-2}v_1$ which occur on C in the order of their indices. Notice that a vertex in $N_C(e_0)^+\cup\{x_1,x_2\}$ has no neighbors in $G-P$; otherwise P is not maximal. Let $v_s\in N_C(x_2)$ and $v_t\in N_C(x_1)$ and $I_s=v_s^+\overrightarrow{C}v_t$ and $I_t=v_t^+\overrightarrow{C}v_s$. If there is a vertex $v_l\in N_{I_s}(v_s^+)^-\cap N_{I_s}(v_t^+)$, then the cycle $v_s^+\overrightarrow{C}v_lv_t^+\overrightarrow{C}v_sx_2x_1v_t\overleftarrow{C}v_l^+v_s^+$ is a desired cycle. See Fig. 4(a). Hence $N_{I_s}(v_s^+)^-\cap N_{I_s}(v_t^+)=\emptyset$. Similarly, we have that

$$N_{I_s}(e_0)^+\cap N_{I_s}(v_t^+)=\emptyset \text{ and } N_{I_s}(v_s^+)^-\cap N_{I_s}(x_1)^+=\emptyset.$$

See Fig. 4(b)–(c). Hence we obtain that

$$|I_s|\geqslant|N_{I_s}(v_s^+)^-|+|N_{I_s}(v_t^+)|+|(N_{I_s}(e_0)\backslash v_t)^+|-|N_{I_s}(v_s^+)^-\cap N_{I_s}(x_2)^+|.$$

Let $L=N_{I_s}(v_s^+)^-\cap N_{I_s}(x_2)^+$. If L is not empty, then for any vertex $v_l\in L$, $v_l^+\notin N_{I_s}(v_s^+)^-$ because G is triangle-free. If $v_l^+v_t^+\in E_G$, then the cycle $v_l^-x_2x_1v_t\overleftarrow{C}v_l^+v_t^+\overrightarrow{C}v_l^-$ is a desired cycle. Since $v_l^+\notin N_C(e_0)^+$,

$$v_l^+\notin N_{I_s}(v_s^+)^-\cup N_{I_s}(v_t^+)\cup N_{I_s}(e_0)^+,$$

and so we deduce that

$$L^+\cap(N_{I_s}(v_s^+)^-\cup N_{I_s}(v_t^+)\cup N_{I_s}(e_0)^+)=\emptyset.$$

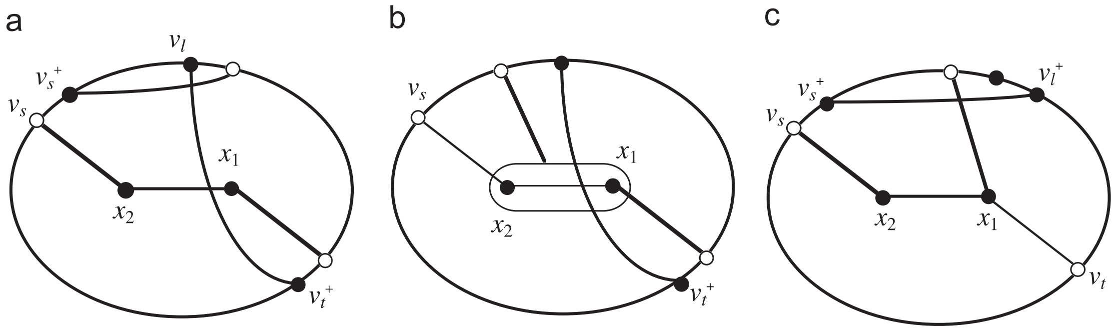

Fig. 4.

Similarly, the vertex v_s^{++} is not contained in $N_{I_s}(v_s^+)^- \cup N_{I_s}(v_t^+) \cup N_{I_s}(e_0)^+$. Therefore we find that

$$\begin{aligned}|I_s| &\geqslant |N_{I_s}(v_s^+)^-| + |N_{I_s}(v_t^+)| + |(N_{I_s}(e_0)\backslash v_t)^+| - |L| + |L^+| + |\{v_s^{++}\}|\\ &\geqslant |N_{I_s}(v_s^+)| + |N_{I_s}(v_t^+)| + |N_{I_s}(e_0)\backslash v_t| + 1\\ &= d_{I_s}(v_s^+) + d_{I_s}(v_t^+) + d_{I_s}(x_1) + d_{I_s}(x_2).\end{aligned}$$

By symmetry, we get $|I_t| \geqslant d_{I_t}(v_s^+) + d_{I_t}(v_t^+) + d_{I_t}(x_1) + d_{I_t}(x_2)$. By (5),

$$\begin{aligned}n - 2 \geqslant |C| = |I_s| + |I_t| &\geqslant d_{I_s}(v_s^+) + d_{I_s}(v_t^+) + d_{I_s}(x_1) + d_{I_s}(x_2)\\ &\quad + d_{I_t}(v_s^+) + d_{I_t}(v_t^+) + d_{I_t}(x_1) + d_{I_t}(x_2)\\ &= d(v_s^+) + d(v_t^+) + (d(x_1) - 1) + (d(x_2) - 1) \geqslant n,\end{aligned}$$

which is a contradiction. This completes the proof of Theorem 3.

3. The proof of Theorem 4

Let $G = (V, E)$ be a triangle-free graph with $\delta \geqslant 2$ and $\sigma_4 \geqslant n + 2$. If G is isomorphic to the exception of Theorem 3, then obviously for any two vertices, there is a cycle containing the specified vertices. By Theorem 3 and the following lemma, it is enough to show that G is connected. A cycle C is called a *swaying* cycle of a subset $S \subseteq V$ if $|C \cap S|$ is maximum over all cycles of G.

Lemma 6. *Let G be a connected graph such that for any path P, there exists a cycle C such that $|P - C| \leqslant 1$. Then for any set S with at most δ vertices, there exists a cycle C such that $S \subset V_C$.*

Proof. Let $S \subseteq V_G$ and let C be a longest swaying cycle of S. Suppose $S - C \neq \emptyset$. For any vertex $x \in S - C$, there is a path Q joining x and C. Let P be a longest path containing $V_{C\cup Q}$. Then there exists a cycle D such that $|P - D| \leqslant 1$. If x has neighbors in $G - C$, then $|P| \geqslant |C| + 2$ and so $|D| \geqslant |C| + 1$. Because $|D \cap S| \geqslant |C \cap S|$, this contradicts the assumption that C is a longest swaying cycle. Hence $N_{G-C}(x) = \emptyset$.

Because $|C \cap S| < \delta$ and $d_C(x) = d(x) \geqslant \delta$, there exist two vertices $v_i, v_j \in N(x)$ such that $v_{i+1} = v_j$ or $v_i^+ \overrightarrow{C} v_j^- \subset C - S$. Hence the cycle $v_i x v_j \overrightarrow{C} v_i$ contains at least $|C \cap S| + 1$ vertices of S. This contradicts the assumption that C is a swaying cycle. $\square$

Before we can prove that G is connected we first need to show the following lemma.

Lemma 7. *Let H be a connected component of a triangle-free graph G. If $|H| \geqslant 3$, then H contains non-adjacent vertices x and y such that $|H| \geqslant \max\{2d(x), 2d(y)\}$.*

Proof. Let $P = u_1u_2 \ldots u_p$ be a longest path of H. If $u_1u_p \notin E_G$, then $|P| \geqslant |N(u_1)| + |N(u_1)^-| + |\{u_p\}| = 2d(u_1)+1$. Hence by symmetry, we have $|H| \geqslant \max\{2d(u_1)+1, 2d(u_p)+1\}$, and so $\{u_1, u_p\}$ is a desired pair. If $u_1u_p \in E_G$, then $u_1u_{p-1} \notin E_G$, and $V_H = V_P$ as P is a longest path. Then, we have

$$|P - u_p| \geqslant |N(u_{p-1})\backslash u_p| + |(N(u_{p-1})\backslash u_p)^+| + |u_1| = 2d(u_{p-1}) - 1.$$

Therefore $|H| \geqslant 2d(u_{p-1})$. As in the above case, we can have $|H| \geqslant 2d(u_1)$, and so $\{u_1, u_{p-1}\}$ is a desired pair. $\square$

By using Lemma 7 we can show that G is indeed connected. This finishes the proof of Theorem 4.

Lemma 8. *Let G be a triangle-free graph with $\delta \geqslant 2$. If $\sigma_4 \geqslant n+1$, then G is connected.*

Proof. Suppose G contains two connected components H_1 and H_2. Then the assumption that G is triangle-free and $\delta \geqslant 2$ implies $H_i \geqslant 3$ for $i = 1, 2$. Therefore there are non-adjacent vertices x_i, y_i in H_i such that $|H_i| \geqslant \max\{2d(x_i), 2d(y_i)\}$ for $i = 1, 2$ by Lemma 7. Hence $d(x_1) + d(y_1) + d(x_2) + d(y_2) \geqslant \sigma_4 \geqslant n + 1$. By symmetry, we may assume $d(x_1) + d(x_2) \geqslant (n+1)/2$. Thus $n \geqslant |H_1| + |H_2| \geqslant 2(d(x_1) + d(x_2)) \geqslant n + 1$, a contradiction. $\square$

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